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Technical Comments

Comment on "Estimation of Fundamental Frequencies of Beams and Plates with Varying Thickness"

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EASTEP[1] has presented a perturbation theory for determining the frequencies and normal modes of beams and plates with variable thickness. All of Eastep's results can be obtained more directly from the well-known classical perturbation theory, first set forth in systematic form by Rayleigh[2] and since then treated with varying degrees of mathematical rigor in many texts on physics, mathematics, and vibration theory.[3-6] Rayleigh's original work is not only general in form but most often leads to the desired results in the most straightforward fashion, as will be illustrated.

The procedure used by Eastep, in which the perturbation process is developed ab initio for each problem utilizing the specific differential (or other) equations of the problem, is also well known. (Such an approach is needed when all of the frequencies and normal modes of the unperturbed problem are not known. See Ref. 7 for an illustrative example of this approach.) In fact, a development exactly equivalent to Eastep's Eqs. (5-10) for a tapered beam was given in the text by Morse (Ref. 5, pp. 164-166) except that the boundary conditions of the beam were left general; to get Eastep's equations, it is only necessary to substitute sine functions as the unperturbed normal modes of a simply supported beam of uniform cross-section in the integrals given by Morse.

The remarkable accuracy shown by Eastep for second-order perturbation theory in the case of the linearly tapered beam and the analogous plate problem is interesting, especially in view of the large thickness perturbations involved (thickness variations up to 90% of the maximum thickness). However, such accuracy is not at all typical of second-order perturbation theory in general and is a result of the special characteristics of the problems considered by Eastep, some of which will be discussed below.

Rayleigh's formulation of perturbation theory[2] begins from the equations of motion of the unperturbed system in terms of generalized coordinates q_n corresponding to the normal modes of vibration. These modes are, by definition, uncoupled from one another; in general, the perturbations couple the uncoupled normal modes. The kinetic energy, $T_0+\delta T$, and, the potential energy, $V_0+\delta V$, of the perturbed system, where the δ refer to the effects of the perturbation, are given in terms of the q_n as

$$T_0+\delta T=\tfrac{1}{2}(a_{1_0}+\delta a_{11})\dot{q}_1^2+\tfrac{1}{2}(a_{2_0}+\delta a_{22})\dot{q}_2^2$$
$$+\ldots+\delta a_{12}\dot{q}_1\dot{q}_2+\delta a_{13}\dot{q}_1\dot{q}_3+\ldots$$
$$V_0+\delta V=\tfrac{1}{2}(c_{1_0}+\delta c_{11})q_1^2+\tfrac{1}{2}(c_{2_0}+\delta c_{22})q_2^2$$
$$+\ldots\delta c_{12}q_1q_2+\delta c_{13}q_1q_3+\ldots \tag{1}$$

In relation to the generalized coordinates, it is apparent here that the a are the generalized masses and the c are the generalized stiffnesses. The equations of motion follow from the substitution of $T_0+\delta T$ and $V_0+\delta V$ into Lagrange's equations, resulting in the nth equation of motion at frequency ω,

$$[-(a_{n_0}+\delta a_{nn})\omega^2+(c_{n_0}+\delta c_{nn})]q_n$$
$$+\sum_{m\neq n}[-\delta a_{nm}\omega^2+\delta c_{nm}]q_m=0 \tag{2}$$

From these equations, Rayleigh's perturbation calculation leads to the perturbation of the nth normal mode to the first order and the nth natural frequency to at least the second order as

$$\frac{q_m}{q_n}=\frac{\omega_{n0}^2\delta a_{nm}-\delta c_{nm}}{a_{m_0}(\omega_{m_0}^2-\omega_{n_0}^2)} \tag{3}$$

and

$$\omega_n^2=\frac{c_{n_0}+\delta c_{nn}}{a_{n_0}+\delta a_{nn}}-\sum_{m\neq n}\frac{(\delta c_{nm}-\omega_{n0}^2\delta a_{nm})^2}{a_{m_0}a_{n_0}(\omega_{m_0}^2-\omega_{n_0}^2)} \tag{4}$$

The development in terms of powers of a perturbation parameter requires expression of the δc_{nm} and δa_{nm} in powers of that parameter and expansion of the denominator of the first terms on the left in a power series as

$$\frac{c_{n0}+\delta c_{nn}}{a_{n_0}+\delta a_{nn}}=\frac{(c_{n0}+\delta c_{nn})}{a_{n_0}}\left[1-\frac{\delta a_{nn}}{a_{n_0}}+\left(\frac{\delta a_{nn}}{a_{n_0}}\right)^2\ldots\right] \tag{5}$$

However, the term prior to expansion can be recognized as the Rayleigh quotient for the perturbed system calculated on the basis of unperturbed normal modes.

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For the problem of vibration of a pin-ended beam with linear taper in height h given by $h=h_0(1-\alpha x)$, the values of the a_{n0}, c_{n0}, δc_{nm}, and δa_{nm} are easily determined from the formulas for kinetic and potential energy of beam vibration. In this case, α is the perturbation parameter. Thus,

$$a_{nm}=\int_0^{\ell} m(x)\varphi_n(x)\varphi_m(x)\,dx \tag{6a}$$

$$c_{nm}=\int_0^{\ell} EI(x)\frac{d^2\varphi_n(x)}{dx^2}\frac{d^2\varphi_m(x)}{dx^2}\,dx \tag{6b}$$

where $m(x)$ is the mass per unit length, $EI(x)$ is the beam bending stiffness, ℓ is the beam length, and $\varphi_k(x)$ is the bending deflection of the beam in the kth normal mode of vibration.

In the case calculated by Eastep, $\varphi_k=\sin k\pi x$ in terms of the coordinate x, which has its origin at one end of the beam and is normalized to unity at the other end. The corresponding unperturbed natural frequencies are given by $\omega=k^2\pi^2(m\ell^4/EI)^{1/2}$. The Rayleigh quotient term of Eq. (4) is then readily found to be

$$\frac{c_{n0}+\delta c_{nn}}{a_{n0}+\delta a_{nn}}=\frac{\pi^4[1-3/2\alpha+3H_2\alpha^2-H_3\alpha^3]}{[1-\alpha/2]} \tag{7}$$

where

$$H_2=2\int_0^1 x^2\sin^2\pi x\,dx=1/3-1/2\pi^2$$

$$H_3=2\int_0^1 x^3\sin^2\pi x\,dx=1/4-3/4\pi^2$$

(These formulas were given by Tricomi,[8] who computed upper and lower bounds for the fundamental frequency of linearly tapered beams.)

However, it may be seen that $2H_3=3H_2-1/2$, so that $1=\alpha/2$ is a factor of the numerator. Thus, to *all* orders of α

$$\frac{c_{n0}+\delta c_{nn}}{a_{n0}+\delta a_{nn}}=\pi^4[1-\alpha+2H_3\alpha^2] \tag{8}$$

This then eliminates contributions of order higher than α^2 from the first term on the right hand side of Eq. (4).

The second term on the right-hand side of Eq. (4) is calculated to the second order in α for $n=1$ as

$$\sum_{m\text{ even}}\frac{(\delta c_{1m}-\omega_{10}^2\delta a_{1m})^2}{a_{m0}a_{10}(\omega_{m0}^2-\omega_{10}^2)}=\frac{64}{\pi^4}\sum_{m\text{ even}}\frac{(1-3m^2)^2,m^2}{(m^4-1)(m^2-1)^4} \tag{9}$$

The computation of δc_{1m} and $\delta\alpha_{1m}$ to order α involves the integral

$$I_{nm}^{(1)}=\int_0^1 x\sin n\pi x\sin m\pi x\,dx=\frac{2}{\pi^2}\frac{nm}{(n^2-m^2)^2} \tag{10}$$

This integral vanishes for $m-n$ odd so that for $n=1$ the I_{nm} is different from zero only for m even. Further, as can be seen from Eq. (9), the terms in the infinite series decrease very rapidly as m increases (e.g., the first term of the series is 0.2617276 and the sum of the first five non-zero terms is 0.2636957). Finally, the "resonance" factor of frequency squared difference in the denominator in Eq. (9) further reduces the effect of terms as m increases, since ω_m^2 varies as m^4. Higher order perturbation calculations involve at least the squares of $I_{nm}^{(2)}$ and $I_{nm}^{(3)}$ (with x^2 and x^3, respectively, replacing x under the integral of Eq. (10), the associated terms in the perturbation series being of order α^4 and α^6, respectively) and double and triple products of $I_{nm}^{(1)}$, $I_{nm}^{(2)}$ and $I_{nn}^{(3)}$, the latter terms being divided by two "resonance" factors of frequency squared difference. Since $I_{12}^{(1)}$ is significantly larger than other $I_{1m}^{(1)}$, it is clear that, for this particular problem, contributions from perturbation orders higher than α^2 will be small. While these characteristics in themselves are no sure evidence that the sum of the infinite series generated by the perturbation process will converge to a sum close to the result for the sum of corrections up to order α^2, they are sufficient to indicate the special nature of this perturbation problem.

Table 1 Fundamental frequency: $\omega(m\ell^4/EI_0)^{1/2}$

Taper ratio α	Perturbation[1]	"Exact"[9]	Perturbation, Eq. (11)
0.5	7.11	7.12	7.12
0.67	5.97	6.00	5.99
0.75	5.36	5.35	5.38
0.90	3.99	3.89	4.05

By adding the two terms of Eq. (9) as calculated above in Eqs. (8) and (9), we obtain the formula for the fundamental natural frequency of a pin-ended beam with linear taper in height represented by taper ratio α as

$$\omega^2=\pi^4\{1-\alpha+0.08432\alpha^2\} \tag{11}$$

accurate to order α^2. A comparison of values from Eq. (11) with the results cited by Eastep is given in Table 1 in terms of the parameter $\omega(m\ell^4/EI_0)^{1/2}$.

The numerical values for vibration frequencies obtained by Conway and Dubil[9] from the solution of the differential equation of motion in terms of Bessel functions have been taken directly from the source and differ slightly from the corresponding values in Ref. 1.

Equation (11) as a very accurate working formula is the main result which can be attributed to Eastep's demonstration of the great accuracy of second order perturbation theory in this specific case. Of course, a similar useful result can be obtained for the analogous case of a simply supported plate with linear thickness variation.

It is interesting to note the observation by Steele[10] that the asymptotic approximation, or the "WKB" method as it is more commonly known in the literature of physics, also leads in the case of pin-ended tapered beams to results of high accuracy even for the fundamental frequency and even more so for the overtones to which, in general, this method would usually be expected to apply.†

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†Reference 10 and its relation to the pin-ended beam problem were called to the author's attention by an editor of the *AIAA Journal*.

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Reply by Author to A. H. Flax

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FLAX has used another perturbation theory, first presented by Rayleigh, for determining the natural frequencies of a beam with variable thickness. He obtains the very interesting result in Eq. (11), which is the fundamental frequency of a linearly tapered beam and is correct to within second order of the taper ratio α. The results obtained by using Eq. (11) do compare quite favorably with those of Ref. 1 and the exact results of Conway and Dubil.[2]

The selection of the simple support conditions and the particular examples (linearly tapered beam and plate) was merely a matter of numerical computation convenience and in no way limits the perturbation method of Ref. 1. Furthermore, it should be mentioned that the results displayed in Ref. 1 were obtained to within second order of a small perturbation parameter for both frequency and mode shape of a beam and plate. In particular, the fundamental mode shape of a linearly tapered plate is given in Fig. 2 and Eq. (19) of Ref. 1.

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Errata

Ionospheric Doppler Sounder for Detection and Prediction of Severe Storms

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PROFESSOR Hung's title was inadvertently omitted from the footnotes for this Technical Note. The footnotes should have read:
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